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# A note on Carlitz $q$ -Bernoulli numbers and polynomials

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## Abstract

In this article, we first aim to give simple proofs of known formulae for the generalized Carlitz  $q$ -Bernoulli polynomials  $\beta_{m,x}(x, q)$  in the  $p$ -adic case by means of a method provided by Kim and then to derive a complex, analytic, two-variable  $q$ - $L$ -function that is a  $q$ -analog of the two-variable  $L$ -function. Using this function, we calculate the values of two-variable  $q$ - $L$ -functions at nonpositive integers and study their properties when  $q$  tends to 1.

**Mathematics Subject Classification (2000):** 11B68; 11S80.

**Keywords:** Carlitz  $q$ -Bernoulli numbers, Carlitz  $q$ -Bernoulli polynomials, Dirichlet  $q$ - $L$ -functions

## 1. Introduction

Let  $p$  be a fixed prime. We denote by  $\mathbb{Z}_p$ ,  $\mathbb{Q}_p$ , and  $\mathbb{C}_p$  the ring of  $p$ -adic integers, the field of  $p$ -adic numbers, and the completion of the algebraic closure of  $\mathbb{Q}_p$ , respectively. Let  $v_p$  be the normalized exponential valuation of  $\mathbb{C}_p$  with  $|p|_p = p^{-v_p(p)} = p^{-1}$ . When one talks of a  $q$ -extension,  $q$  can be variously considered as an indeterminate, a complex number  $q \in \mathbb{C}$ , or a  $p$ -adic number  $q \in \mathbb{C}_p$ . If  $q \in \mathbb{C}$ , one normally assumes  $|q| < 1$ . If  $q \in \mathbb{C}_p$ , one normally assumes  $|1 - q|_p < p^{-1/(p-1)}$ , so that  $q^x = \exp(x \log_p q)$  for  $|x|_p \leq 1$ .

Let  $d$  be a fixed positive integer. Let

$$X = X_d = \varprojlim_{\mathbb{N}} (\mathbb{Z}/dp^N \mathbb{Z}), \quad X_1 = \mathbb{Z}_p,$$
$$X^* = \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \quad (1.1)$$

$$a + dp^N \mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where  $a \in \mathbb{Z}$  lies in  $0 \leq a < dp^N$ . We use the following notation:

$$[x]_q = \frac{1 - q^x}{1 - q}. \quad (1.2)$$

Hence  $\lim_{q \rightarrow 1} [x]_q = x$  for any  $x \in \mathbb{C}$  in the complex case and any  $x$  with  $|x|_p \leq 1$  in the present  $p$ -adic case. This is the hallmark of a  $q$ -analog: The limit as  $q \rightarrow 1$  recovers the classical object.

In 1937, Vandiver [1] and, in 1941, Carlitz [2] discussed generalized Bernoulli and Euler numbers. Since that time, many authors have studied these and other related

subjects (see, e.g., [3-6]). The final breakthrough came in the 1948 article by Carlitz [7]. He defined inductively new  $q$ -Bernoulli numbers  $\beta_m = \beta_m(q)$  by

$$\beta_0(q) = 1, \quad q(q\beta(q) + 1)^m - \beta_m(q) = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{if } m > 1, \end{cases} \quad (1.3)$$

with the usual convention of  $\beta^i$  by  $\beta_i$ . The  $q$ -Bernoulli polynomials are defined by

$$\beta_m(x, q) = (q^x \beta(q) + [x]_q)^m = \sum_{i=0}^m \binom{m}{i} \beta_i(q) q^{ix} [x]_q^{m-i}. \quad (1.4)$$

In 1954, Carlitz [8] generalized a result of Frobenius [3] and showed many of the properties of the  $q$ -Bernoulli numbers  $\beta_m(q)$ . In 1964, Carlitz [9] extended the Bernoulli, Eulerian, and Euler numbers and corresponding polynomials as a formal Dirichlet series. In what follows, we shall call them the Carlitz  $q$ -Bernoulli numbers and polynomials.

Some properties of Carlitz  $q$ -Bernoulli numbers  $\beta_m(q)$  were investigated by various authors. In [10], Koblitz constructed a  $q$ -analog of  $p$ -adic  $L$ -functions and suggested two questions. Question (1) was solved by Satoh [11]. He constructed a complex analytic  $q$ - $L$ -series that is a  $q$ -analog of Dirichlet  $L$ -function and interpolates Carlitz  $q$ -Bernoulli numbers, which is an answer to Koblitz's question. By using a  $q$ -analog of the  $p$ -adic Haar distribution (see (1.6) below), Kim [12] answered part of Koblitz's question (2) and constructed  $q$ -analogs of the  $p$ -adic log gamma functions  $G_{p,q}(x)$  on  $\mathbb{C}_p \setminus \mathbb{Z}_p$ .

In [11], Satoh constructed the generating function of the Carlitz  $q$ -Bernoulli numbers  $F_q(t)$  in  $\mathbb{C}$  which is given by

$$F_q(t) = \sum_{m=0}^{\infty} q^m e^{[m]_q t} (1 - q - q^m t) = \sum_{m=0}^{\infty} \beta_m(q) \frac{t^m}{m!}, \quad (1.5)$$

where  $q$  is a complex number with  $0 < |q| < 1$ . He could not explicitly determine  $F_q(t)$  in  $\mathbb{C}_p$ , see [11, p.347].

In [12], Kim defined the  $q$ -analog of the  $p$ -adic Haar distribution  $\mu_{\text{Haar}}(a + p^N \mathbb{Z}_p) = 1/p^N$  by

$$\mu_q(a + p^N \mathbb{Z}_p) = \frac{q^a}{[p^N]_q}. \quad (1.6)$$

Using this distribution, he proved that the Carlitz  $q$ -Bernoulli numbers  $\beta_m(q)$  can be represented as the  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  by  $\mu_q$ , that is,

$$\beta_m(q) = \int_{\mathbb{Z}_p} [a]_q^m d\mu_q(a), \quad (1.7)$$

and found the following explicit formula

$$\beta_m(q) = \frac{1}{(q-1)^m} \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} \frac{i+1}{[i+1]_q}, \quad (1.8)$$

where  $m \geq 0$  and  $q \in \mathbb{C}_p$  with  $0 < |1-q|_p < p^{-\frac{1}{p-1}}$ .

Recently, Kim and Rim [13] constructed the generating function of the Carlitz  $q$ -Bernoulli numbers  $F_q(t)$  in  $\mathbb{C}_p$  :

$$F_q(t) = e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \frac{j+1}{[j+1]_q} (-1)^j \left( \frac{1}{1-q} \right)^j \frac{t^j}{j!}, \quad (1.9)$$

where  $q \in \mathbb{C}_p$  with  $0 < |1 - q|_p < p^{-\frac{1}{p-1}}$ .

This article is organized as follows.

In Section 2, we consider the generalized Carlitz  $q$ -Bernoulli polynomials in the  $p$ -adic case by means of a method provided by Kim. We obtain the generating functions of the generalized Carlitz  $q$ -Bernoulli polynomials. We shall provide some basic formulas for the generalized Carlitz  $q$ -Bernoulli polynomials which will be used to prove the main results of this article.

In Section 3, we construct the complex, analytic, two-variable  $q$ - $L$ -function that is a  $q$ -analog of the two-variable  $L$ -function. Using this function, we calculate the values of two-variable  $q$ - $L$ -functions at nonpositive integers and study their properties when  $q$  tends to 1.

## 2. Generalized Carlitz $q$ -Bernoulli polynomials in the $p$ -adic (and complex) case

For any uniformly differentiable function  $f: \mathbb{Z}_p \rightarrow \mathbb{C}_p$ , the  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  is defined to be the limit  $\frac{1}{[p^N]_q} \sum_{a=0}^{p^N-1} f(a)q^a$  as  $N \rightarrow \infty$ . The uniform differentiability guarantees the limit exists. Kim [12,14-16] introduced this construction, denoted  $I_q(f)$ , where  $|1 - q|_p < p^{-1/(p-1)}$ .

The construction of  $I_q(f)$  makes sense for many  $q$  in  $\mathbb{C}_p$  with the weaker condition  $|1 - q|_p < 1$ . Indeed, when  $|1 - q|_p < 1$  the function  $q^x$  is uniformly differentiable and the space of uniformly differentiable functions  $\mathbb{Z}_p \rightarrow \mathbb{C}_p$  is closed under multiplication, so we can make sense of its  $p$ -adic  $q$ -integral  $I_q(f)$  for  $|1 - q|_p < 1$ .

**Lemma 2.1.** For  $q \in \mathbb{C}_p$  with  $0 < |1 - q|_p < 1$  and  $x \in \mathbb{Z}_p$ , we have

$$\lim_{N \rightarrow \infty} \frac{1}{1 - q^{p^N}} \sum_{a=0}^{p^N-1} q^{ax} = \frac{x}{1 - q^x}.$$

*Proof.* We assume that  $q \in \mathbb{C}_p$  satisfies the condition  $0 < |1 - q|_p < 1$ . Then it is known that

$$q^x = \sum_{m=0}^{\infty} \binom{x}{m} (q - 1)^m$$

for any  $x \in \mathbb{Z}_p$  (see [[17], Lemma 3.1 (iii)]). Therefore, we obtain

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{1 - q^{p^N}} \sum_{a=0}^{p^N-1} q^{ax} &= \frac{1}{1 - q^x} \lim_{N \rightarrow \infty} \frac{(q^{p^N})^x - 1}{q^{p^N} - 1} \\ &= \frac{1}{1 - q^x} \lim_{N \rightarrow \infty} \frac{\sum_{m=1}^{\infty} \binom{x}{m} (q^{p^N} - 1)^m}{q^{p^N} - 1} \\ &= \frac{1}{1 - q^x} \lim_{N \rightarrow \infty} \sum_{m=0}^{\infty} \binom{x}{m+1} (q^{p^N} - 1)^m \\ &= \frac{x}{1 - q^x}. \end{aligned}$$

This completes the proof.

**Definition 2.2** ([12, §2, p. 323]). Let  $\chi$  be a primitive Dirichlet character with conductor  $d \in \mathbb{N}$  and let  $x \in \mathbb{Z}_p$ . For  $q \in \mathbb{C}_p$  with  $0 < |1 - q|_p < 1$  and an integer  $m \geq 0$ , the generalized Carlitz  $q$ -Bernoulli polynomials  $\beta_{m,\chi}(x, q)$  are defined by

$$\begin{aligned}\beta_{m,\chi}(x, q) &= \int_{\mathbb{X}} \chi(a)[x+a]_q^m d\mu_q(a) \\ &= \lim_{N \rightarrow \infty} \frac{1}{[dp^N]_q} \sum_{a=0}^{dp^N-1} \chi(a)[x+a]_q^m q^a.\end{aligned}\quad (2.1)$$

*Remark 2.3.* If  $\chi = \chi^0$ , the trivial character and  $x = 0$ , then (2.1) reduces to (1.7) since  $d = 1$ . In particular, Kim [12] defined a class of  $p$ -adic interpolation functions  $G_{p,q}(x)$  of the Carlitz  $q$ -Bernoulli numbers  $\beta_m(q)$  and gave several interesting applications of these functions.

By Lemma 2.1, we can prove the following explicit formula of  $\beta_{m,\chi}(x, q)$  in  $\mathbb{C}_p$ .

**Proposition 2.4.** For  $q \in \mathbb{C}_p$  with  $0 < |1 - q|_p < 1$  and an integer  $m \geq 0$ , we have

$$\beta_{m,\chi}(x, q) = \frac{1}{(1-q)^m} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^m \binom{m}{i} (-1)^i q^{i(x+k)} \frac{i+1}{[d(i+1)]_q}.$$

*Proof.* For  $m \geq 0$ , (2.1) implies

$$\begin{aligned}\beta_{m,\chi}(x, q) &= \lim_{N \rightarrow \infty} \frac{1}{[d]_q} \frac{1}{[p^N]_{q^d}} \sum_{k=0}^{d-1} \sum_{a=0}^{p^N-1} \chi(k+da)[x+k+da]_q^m q^{k+da} \\ &= \lim_{N \rightarrow \infty} \frac{1}{(1-q)^{m-1}} \sum_{k=0}^{d-1} \chi(k) \frac{q^k}{1-q^{dp^N}} \sum_{q=0}^{p^N-1} (1-q^{x+k+da})^m q^{da} \\ &= \frac{1}{(1-q)^{m-1}} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^m \binom{m}{i} (-1)^i q^{i(x+k)} \\ &\quad \times \lim_{N \rightarrow \infty} \frac{1}{1-(q^d)^{p^N}} \sum_{a=0}^{p^N-1} (q^d)^{a(i+1)} \\ &= \frac{1}{(1-q)^{m-1}} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^m \binom{m}{i} (-1)^i q^{i(x+k)} \frac{i+1}{1-q^{d(i+1)}} \\ &\quad \text{(where we use Lemma 2.1).}\end{aligned}$$

This completes the proof.

*Remark 2.5.* We note here that similar expressions to those of Proposition 2.4 with  $\chi = \chi^0$  are given by Kamano [[18], Proposition 2.6] and Kim [12, §2]. Also, Ryoo et al. [19, Theorem 4] gave the explicit formula of  $\beta_{m,\chi}(0, q)$  in  $\mathbb{C}$  for  $m \geq 0$ .

**Lemma 2.6.** Let  $\chi$  be a primitive Dirichlet character with conductor  $d \in \mathbb{N}$ . Then for  $q \in \mathbb{C}$  with  $|q| < 1$ ,

$$\sum_{m=0}^{\infty} \chi(m) q^{mx} = \frac{1}{1-q^d} \sum_{k=0}^{d-1} \chi(k) q^{kx}.$$

*Proof.* If we write  $m = ad + k$ , where  $0 \leq k \leq d-1$  and  $a = 0, 1, 2, \dots$ , we have the desired result.

We now consider the case:

$$q \in \overline{\mathbb{Q}} \cap \mathbb{C}_p, \quad 0 < |q| < 1, \quad 0 < |1 - q|_p < 1. \quad (2.2)$$

For instance, if we set

$$q = \frac{1}{1 - pz} \in \overline{\mathbb{Q}} \cap \mathbb{C}_p$$

for each  $z \neq 0 \in \mathbb{Z}$  and  $p > 3$ , we find  $0 < |q| < 1$ ,  $0 < |1 - q|_p < 1$ .

Let  $F_{q,\chi}(t, x)$  be the generating function of  $\beta_{m,\chi}(x, q)$  defined in Definition 2.2. From Proposition 2.4, we have

$$\begin{aligned} F_{q,\chi}(t, x) &= \sum_{m=0}^{\infty} \beta_{m,\chi}(x, q) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left( \frac{1}{(1-q)^m} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^m \binom{m}{i} (-1)^i q^{i(x+k)} \frac{i+1}{[d(i+1)]_q} \right) \frac{t^m}{m!} \\ &= P_{q,\chi}(t, x) + Q_{q,\chi}(t, x), \end{aligned} \quad (2.3)$$

where

$$P_{q,\chi}(t, x) = \sum_{m=0}^{\infty} \frac{1}{(1-q)^m} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^m \binom{m}{i} (-1)^i q^{i(x+k)} \frac{i}{[d(i+1)]_q} \frac{t^m}{m!}$$

and

$$Q_{q,\chi}(t, x) = \sum_{m=0}^{\infty} \frac{1}{(1-q)^m} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^m \binom{m}{i} (-1)^i q^{i(x+k)} \frac{1}{[d(i+1)]_q} \frac{t^m}{m!}.$$

Then, noting that

$$e^{\frac{t}{1-q}} = \sum_{i=0}^{\infty} (-1)^i (q-1)^{-i} \frac{t^i}{i!},$$

we see that

$$\begin{aligned} P_{q,\chi}(t, x) &= \sum_{m=0}^{\infty} \frac{1}{(1-q)^m} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^m \binom{m}{i} (-1)^i q^{i(x+k)} \frac{i}{[d(i+1)]_q} \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \frac{t^n}{n!} \sum_{j=0}^{\infty} \frac{1}{(q-1)^j} \sum_{k=0}^{d-1} \chi(k) q^{j(x+k)+k} \frac{j}{[d(j+1)]_q} \frac{t^j}{j!} \\ &= e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \left( \frac{1}{q-1} \right)^j \sum_{k=0}^{d-1} \chi(k) q^{j(x+k)+k} \frac{j}{[d(j+1)]_q} \frac{t^j}{j!}. \end{aligned} \quad (2.4)$$

Moreover, (2.4) now becomes

$$\begin{aligned}
 P_{q,\chi}(t, x) &= e^{\frac{t}{1-q}} \sum_{j=1}^{\infty} \left( \frac{1}{q-1} \right)^j \sum_{k=0}^{d-1} \chi(k) q^{j(x+k)+k} \frac{1}{[d(j+1)]_q} \frac{t^j}{(j-1)!} \\
 &= e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \left( \frac{1}{q-1} \right)^j q^{(j+1)x} \sum_{k=0}^{d-1} \chi(k) \frac{q^{k(j+2)}}{q^{d(j+2)} - 1} \frac{t^{j+1}}{j!} \\
 &= -te^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \left( \frac{1}{q-1} \right)^j q^{(j+1)x} \sum_{n=0}^{\infty} \chi(n) q^{n(j+2)} \frac{t^j}{j!} \\
 &\quad \text{(where we use Lemma 2.6)} \\
 &= -te^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \chi(n) q^{x+2n} \sum_{j=0}^{\infty} \left( \frac{-q^{n+x}}{1-q} \right)^j \frac{t^j}{j!} \\
 &= -te^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \chi(x) q^{x+2n} e^{\frac{(-q^{n+x})t}{1-q}} \\
 &= -t \sum_{n=0}^{\infty} \chi(x) q^{x+2n} e^{[n+x]_q t}
 \end{aligned} \tag{2.5}$$

(cf. [13,16,20]). Similar arguments apply to the case  $Q_{q,\chi}(t, x)$ . We can rewrite

$$Q_{q,\chi}(t, x) = e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \left( \frac{1}{q-1} \right)^j \sum_{k=0}^{d-1} \chi(k) q^{j(x+k)+k} \frac{1}{[d(j+1)]_q} \frac{t^j}{j!} \tag{2.6}$$

and

$$Q_{q,\chi}(t, x) = (1-q) \sum_{n=0}^{\infty} \chi(n) q^n e^{[n+x]_q t}. \tag{2.7}$$

Then, by (2.4), (2.5), (2.6), and (2.7), we have the following theorem.

**Theorem 2.7.** *Let  $q \in \overline{\mathbb{Q}} \cap \mathbb{C}_p$ ,  $0 < |q| < 1$ ,  $0 < |1-q|_p < 1$ . Then the generalized Carlitz  $q$ -Bernoulli polynomials  $\beta_{m,\chi}(x, q)$  for  $m \leq 0$  is given by equating the coefficients of powers of  $t$  in the following generating function:*

$$\begin{aligned}
 F_{q,\chi}(t, x) &= e^{\frac{t}{1-q}} \sum_{j=0}^{\infty} \left( \frac{1}{q-1} \right)^{j-1} \sum_{k=0}^{d-1} \chi(k) q^{j(x+k)+k} \frac{j+1}{q^{d(j+1)} - 1} \frac{t^j}{j!} \\
 &= \sum_{n=0}^{\infty} \chi(n) q^n e^{[n+x]_q t} (1-q-q^{n+x}t).
 \end{aligned} \tag{2.8}$$

*Remark 2.8.* If  $\chi = \chi_0$ , the trivial character, and  $x = 0$ , (2.8) reduces to (1.5).

### 3. $q$ -analog of the two-variable $L$ -function (in $\mathbb{C}$ )

From Theorem 2.7, for  $k \geq 0$ , we obtain the following

$$\begin{aligned}
 \beta_{k,\chi}(x, q) &= \left( \frac{d}{dt} \right)^k F_{q,\chi}(t, x) \Big|_{t=0} \\
 &= (1-q) \sum_{m=0}^{\infty} \chi(m) q^m [m+x]_q^k - k \sum_{m=0}^{\infty} \chi(m) q^{x+2m} [m+x]_q^{k-1}.
 \end{aligned} \tag{3.1}$$

Hence we can define a  $q$ -analog of the  $L$ -function as follows:

**Definition 3.1.** Suppose that  $\chi$  is a primitive Dirichlet character with conductor  $d \in \mathbb{N}$ . Let  $q$  be a complex number with  $0 < |q| < 1$ , and let  $L_q(s, x, \chi)$  be a function of two-variable  $(s, x) \in \mathbb{C} \times \mathbb{R}$  defined by

$$L_q(s, x, \chi) = \frac{1-q}{s-1} \sum_{m=0}^{\infty} \frac{\chi(m)q^m}{[m+x]_q^{s-1}} + \sum_{m=0}^{\infty} \frac{\chi(m)q^{m+2x}}{[m+x]_q^s} \quad (3.2)$$

for  $0 < x \leq 1$  (cf. [11,13,14,21-25]).

In particular, the two-variable function  $L_q(s, x, \chi)$  is a generalization of the one-variable  $L_q(s, \chi)$  of Satoh [11], yielding the one-variable function when the second variable vanishes.

**Proposition 3.2.** For  $k \in \mathbb{Z}$ ,  $k \geq 1$ , the limiting value  $\lim_{s \rightarrow k} L_q(1-s, x, \chi) = L_q(1-k, x, \chi)$  exists and is given explicitly by

$$L_q(1-k, x, \chi) = -\frac{1}{k} \beta_{k,\chi}(x, q).$$

*Proof.* The proof is clear by Proposition 2.4, Theorem 2.7 and (3.1).

The formula of Proposition 3.2 is slight extension of the result in [19] and [11, Theorem 2].

**Theorem 3.3.** For any positive integer  $k$ , we have

$$\begin{aligned} \lim_{q \rightarrow 1} \beta_{k,\chi}(x, q) &= \lim_{q \rightarrow 1} \frac{1}{(1-q)^m} \sum_{k=0}^{d-1} \chi(k) q^k \sum_{i=0}^m \binom{m}{i} (-1)^i q^{i(x+k)} \frac{i+1}{[d(i+1)]_q} \\ &= B_{k,\chi}(x), \end{aligned}$$

where the  $B_{k,\chi}(x)$  are the  $k$ th generalized Bernoulli polynomials.

*Proof.* We follow the proof in [[26], Theorem 1] motivated by the study of a simple  $q$ -analog of the Riemann zeta function. Recall that the ordinary Bernoulli polynomials  $B_k(x)$  are defined by

$$\frac{i}{q^i - 1} q^{ix} = \frac{1}{\log q} \frac{i \log q}{e^{i \log q} - 1} e^{x(i \log q)} = \frac{1}{\log q} \sum_{k=0}^{\infty} B_k(x) i^k \frac{(\log q)^k}{k!}, \quad (3.3)$$

where it is noted that in this instance, the notation  $B_k(x)$  is used to replace  $B^k(x)$  symbolically. For each  $m \geq 1$ , let

$$(e^t - 1)^m = \sum_{k=0}^{\infty} d_k^{(m)} \frac{t^k}{k!}. \quad (3.4)$$

Note that

$$(e^t - 1)^m = \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} e^{it} = \sum_{k=0}^{\infty} \left( \sum_{i=0}^m \binom{m}{i} (-1)^{m-i} i^k \right) \frac{t^k}{k!}. \quad (3.5)$$

From (3.4) and (3.5), we obtain

$$d_k^{(m)} = \begin{cases} \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} i^k, & m \leq k \\ 0, & 0 \leq k < m. \end{cases} \quad (3.6)$$

It is also clear from the definition that  $d_0^{(0)} = 1$ ,  $d_k^{(0)} = 0$  and  $d_k^{(k)} = k!$  for  $k \in \mathbb{N}$ . From (2.3), (3.3), and (3.6), we obtain

$$\begin{aligned}\beta_{m,\chi}(x, q) &= \frac{q^{-x}}{(q-1)^m} \sum_{k=0}^{d-1} \chi(k) q^{k+x} \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} q^{i(k+x)} \frac{i+1}{[d(i+1)]_q} \\ &= \frac{q^{-x}}{(q-1)^{m-1}} \sum_{k=0}^{d-1} \chi(k) \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} \\ &\quad \times e^{d(i+1)\log q \frac{(k+x)}{d}} \frac{d(i+1)\log q}{e^{d(i+1)\log q-1}} \frac{1}{d\log q} \\ &= \frac{q^{-x}}{(q-1)^{m-1}} \sum_{n=0}^{\infty} \left( \sum_{i=0}^m (-1)^{m-i} \binom{m}{i} (i+1)^n \right) \\ &\quad \times d^{n-1} \sum_{k=0}^{d-1} \chi(k) B_n \left( \frac{k+x}{d} \right) \frac{(\log q)^{n-1}}{n!} \\ &= q^{-x} \frac{(\log q)^{m-1}}{(q-1)^{m-1}} d^{m-1} \sum_{k=0}^{d-1} \chi(k) B_m \left( \frac{k+x}{d} \right) \\ &\quad + q^{-x} \sum_{\sigma=1}^{\infty} \sum_{i=0}^{\sigma} \binom{m+\sigma}{i} d_{m+\sigma-i}^{(m)} \frac{1}{(m+\sigma)!} \frac{(\log q)^{m+\sigma-1}}{(q-1)^{m-1}} \\ &\quad \times d^{m+\sigma-1} \sum_{k=0}^{d-1} \chi(k) B_{m+\sigma} \left( \frac{k+x}{d} \right).\end{aligned}$$

Then, because

$$\log q = \log(1 + (q-1)) = (q-1) - \frac{(q-1)^2}{2} + \dots = (q-1) + O((q-1)^2)$$

as  $q \rightarrow 1$ , we find

$$\lim_{q \rightarrow 1} \frac{(\log q)^{m+\sigma-1}}{(q-1)^{m-1}} = \begin{cases} 1, & \sigma = 0 \\ 0, & \sigma \geq 1, \end{cases}$$

so

$$\lim_{q \rightarrow 1} \beta_{m,\chi}(x, q) = d^{m-1} \sum_{k=0}^{d-1} \chi(k) B_m \left( \frac{k+x}{d} \right) = B_{m,\chi}(x),$$

where the  $B_{m,\chi}(x)$  are the  $m$ th generalized Bernoulli polynomials (e.g., [14,19]). This completes the proof.

**Corollary 3.4.** *For any positive integer  $k$ , we have*

$$\lim_{q \rightarrow 1} L_q(1-k, x, \chi) = -\frac{1}{k} B_{k,x}(x).$$

**Remark 3.5.** The formula of Theorem 3.3 is slight extension of the result in [[26], Theorem 1].

**Remark 3.6.** From Theorem 2.7, the generalized Bernoulli polynomials  $B_{m,\chi}(x)$  are defined by means of the following generating function [[27], p. 8]



$$\begin{aligned} F_{\chi}(t, x) &:= \lim_{q \rightarrow 1} F_{q, \chi}(t, x) \\ &= -t \sum_{a=1}^d \sum_{l=0}^{\infty} \chi(a + dl) e^{(a+dl)t} e^{xt} \\ &= \sum_{a=1}^d \frac{\chi(a) t e^{(a+x)t}}{e^{dt} - 1} \\ &= \sum_{m=0}^{\infty} B_{m, \chi}(x) \frac{t^m}{m!}. \end{aligned}$$

**Remark 3.7.** If we substitute  $\chi = \chi^0$ , the trivial character, in Definition 3.1 and Corollary 3.4, we can also define a  $q$ -analog of the Hurwitz zeta function

$$\zeta(s, x) = \sum_{m=0}^{\infty} \frac{1}{(m+x)^s}$$

by

$$\zeta_q(s, x) = L_q(s, x, \chi^0) = \frac{1-q}{s-1} \sum_{m=0}^{\infty} \frac{q^{m+x}}{[m+x]_q^{s-1}} + \sum_{m=0}^{\infty} \frac{q^{2(m+x)}}{[m+x]_q^s}$$

and obtain the identity

$$\lim_{q \rightarrow 1} \zeta_q(s, x) = \zeta(s, x)$$

for all  $s \neq 1$ , as well as the formula

$$\lim_{q \rightarrow 1} \zeta_q(1-k, x) = -\frac{1}{k} B_k(x)$$

for integers  $k \geq 1$  (cf. [11,13,19,22,24,25]).

#### Acknowledgements

This work was supported by the Kyungnam University Foundation Grant, 2012.

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#### Authors' contributions

The authors have equal contributions to each part of this paper. All authors read and approved the final manuscript.

#### Competing interests

The authors declare that they have no competing interests.

Received: 21 December 2011 Accepted: 13 April 2012 Published: 13 April 2012

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doi:10.1186/1687-1847-2012-44

**Cite this article as:** Kim and Kim: A note on Carlitz  $q$ -Bernoulli numbers and polynomials. *Advances in Difference Equations* 2012 **2012**:44.

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